A Multi-Input Multi-Output Delayed Feedback Controller for Stabilizing Periodic Solutions of the Lorenz System

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Abstract. In this paper the idea of harmonic balance method is used in a new framework to analyze and predict the periodic solutions of the Lorenz system. An analytic equation has been derived for these predicted limit cycles for the first time. The proposed method is fairly straightforward avoiding complicated calculations. A multi-input multi-output Delayed Feedback Controller (DFC) is designed and implemented for stabilizing unstable periodic solutions of the Lorenz system. All previous works done on stabilization of periodic solutions of this system, using a simple DFC (without adding a new dynamic to the system) were unsuccessful. Choosing an appropriate signal to use in the delayed feedback loop and an appropriate point for introducing the control signal are very important tasks in DFC implementation. Considering these facts, we overcome the mentioned problem by choosing the third state variable of the Lorenz system that to our knowledge has not been used before, in the delayed feedback loop and introducing the control signal to the system in a different point from previous works. Our proposed controller is also able to stabilize the equilibrium points (EPs) of the system. The stability analysis is also done.
Keywords: Lorenz system, delay feedback control, harmonic balance.

1 Introduction

DFC is an efficient method of chaos control, which stabilizes Unstable Periodic Orbits (UPO) embedded in a chaotic attractor. In 1994 researchers found out that DFC is not able to stabilize systems with odd number of Floquet exponents. In other words, they thought it is impossible to stabilize any UPOs with odd-number of real characteristic multipliers greater than unity [1–3]. So they tried to overcome this limitation. In [4] authors used an expanded DFC. In [5] it was shown that this stable controller can not overcome all the DFCs limitations. Since they thought these limitations were due to the odd number of positive Floquet exponents, in later studies, researchers tried to solve this problem by adding an unstable term to change the total number of real-positive Floquet exponent to an even number [6,7]. Another method was using some different values of delay in the feedback to increase the controllers degrees of freedom [8,9].

Variety of methods were suggested to eliminate this problem till 2007, but in[10] it was shown that, this limitation that scientists were trying to overcome for more than fifteen years did not exist at all and theoretical analysis and simulations confirmed this fact too [11]. One of the systems that was thought it can not be stabilized with the use of the DFC, due to the "odd-number limitation" was the Lorenz system. Several studies were done to avoid this limitation (see [7,12,13]. In all these studies, it was tried to avoid the limitation by introducing an unstable degree of freedom in a feedback loop to change the number of unstable torsion-free modes to an even number and the control signal was just applied to one of the state variables (the second one).

In this paper we use a simple DFC to stabilize an unstable periodic solution of the Lorenz system. The key idea of our work is using the third state variable of the Lorenz system in the control loop and introducing the controller to both the second and the third state equations (see Eq.1). The next section is devoted to the open loop analysis of the system. Using the Harmonic Balance (HB) idea, the analytical predicted periodic solutions of the system have been derived and their stability features have been studied. In section 3 an analysis is done to predict the chaotic dynamics and finally in section 4, a MIMO DFC is used to stabilize an unstable periodic solution of the system. Also it has shown that this control structure can be used for stabilization of the EPs of the system.

2 The open loop analysis

Consider the following classical Lorenz chaotic system

$$\begin{cases} \dot{x} = -\sigma x + \sigma y \\ \dot{y} = \rho x - y - zx \\ \dot{z} = -\mu z + xy \end{cases} \tag{1}$$

The Lorenz equations have three parameters σ, μ and ρ. To simplify matters, most researchers have kept $\sigma = 10$ and $\mu = 8/3$ while varying ρ . As shown in [14] by assuming $f = x$ the system equations can be rewritten in the following form

$$\begin{cases} \frac{1}{\sigma} + (1 + \frac{1}{\sigma})\dot{f} + (1 - \rho) + fz = 0 \\ \dot{z} = -\mu z + f(\frac{1}{\sigma}\dot{f} + f) \end{cases} \tag{2}$$

Eq.2 puts in evidence the feedback structure of the system, as shown in Fig.1 where a linear subsystem is connected to a nonlinear one. Due to the existence of the dynamical term $1/(s+\mu)$ in the nonlinear subsystem of Fig.1, it may be difficult to use the general approach originally proposed by Tesi

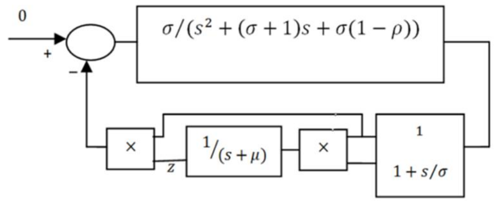

Fig. 1. The feedback structure of the Lorenz system.

and Genesio in 1992 to find the periodic solutions of the system. So in this paper we use the idea of the well-known HB method in the following simple manner.

2.1 Periodic solutions

At the first step, it is assumed that these steady solutions can be approximated as

$$x = A + B\cos\omega t \tag{3}$$

Substituting Eq.3 in the first state equation of Eq.1 results in

$$y = \frac{1}{\sigma}\dot{x} + x = A + B\cos(\omega t) - \frac{1}{\sigma}B\omega\sin(\omega t) \tag{4}$$

Then by substituting Eq.3 and Eq.4 in the third state equation of Eq.1, eliminating the second harmonics which appear, and ignoring the transient solution, we conclude the steady solution of z as

$$z = \frac{3}{80}(10A^2 + 5B^2 + \frac{8AB(160+3\omega^2)\cos(\omega t)}{64+9\omega^2} + \frac{416AB\omega\sin(\omega t)}{64+9\omega^2}) \tag{5}$$

After substituting Eq.3 and Eq.5 in the second state equation of Eq.1 and doing similar calculations, we obtain an expression for y that should be equal to Eq.4. Equalizing the related coefficients results in the following three equations for A^2, B^2 and ω^2 in terms of ρ

$$\begin{aligned} A^2 = \frac{-2}{1561512117}(&527280000\sqrt{3}\rho m - 3845514915\rho + 4056000\sqrt{3}m^3 \\ &- 12168000\sqrt{3}\rho^2 m + 106221667541 - 9892726305\sqrt{3}m) \end{aligned} \tag{6}$$

$$\begin{aligned} B^2 = \frac{4}{425866941}(&21884475789 - 2258100\sqrt{3}m^3 + 4498028807\sqrt{3}m \\ &- 319929891\rho - 293553000\sqrt{3}\rho m + 6774300\sqrt{3}\rho^2 m) \end{aligned} \tag{7}$$

$$\omega^2 = \frac{40}{201}(-1113 + 39\rho + 13\sqrt{3}m) \tag{8}$$

where $m = \sqrt{2123 - 130\rho + 3\rho^2}$. The solution of the problem is possible, when the termsA^2, B^2 and ω^2 are real and positive. Therefore, we can derive the domain of parameter space where there are admissible solutions. The domain of existence is

$$7.7693 \leq \rho \leq 24.7368 \tag{9}$$

The obtained results on periodic solutions (Predicted Limit Cycles) are approximate, due to the first harmonic analysis carried out on the system. The reliability of a PLC is based on a strong attenuation (filtering hypothesis) of the higher frequency components 2ω, 3ω ,$\cdots$ along the loop.

2.2 Stability analysis

The system EPs are: $C^{\pm} = (\pm\sqrt{(\mu(\rho-1))}, \pm\sqrt{(\mu(\rho-1))}, \rho - 1)$ that exist for $\rho > 1$ and $C^0 = (0, 0, 0)$. In this region C^0 is a saddle and $C^{\pm}$ are symmetric stable fixed points. For $0 < \rho \leq 1$ there exist just C^0 which is a stable node. We use the Loeb criterion to check the stability features of PLCs. according to this criteria, in case the PLC be stable, the following inequality will be true[14]:

$$\frac{\Delta\omega}{\Delta B} = \frac{\delta\omega/\delta\rho}{\delta B/\delta\rho} \leq 0 \tag{10}$$

That is true in our case for $\rho \geq 15.1$.

3 The chaotic dynamic prediction

In this section the famous phenomenon of Homoclinic Orbit (HO) which is one of the main routes to chaos in the most dynamic systems has been analyzed and an approximate region of parameter space is derived in which this phenomenon may occur. As stated in [15], the HO conditions includes the existence of a stable PLC and a saddle type EP (different from that generating the PLC) and the interaction between PLC and EP as $B \geq |E - A|$, where E denotes the mentioned saddle EP and is equal to zero in the Lorenz case. This inequality is valid for $7.77 \leq \rho \leq 15.06$. Considering the regions of PLC existence (Eq.9), PLC stabilization ($\rho \geq 15.1$) and the region in which the interaction condition is satisfied ($7.77 \leq \rho \leq 15.06$), we predict that the HO phenomenon may occur at some values around $\rho = 15$. Therefore the Lorenz system may show chaotic behaviors. Numerical solutions show that Homoclinic bifurcation occurs at $\rho = 13.962$ which is near the predicted value.

4 Chaos control

It is obvious that this system exhibits a chaotic behavior in some regions of its parameter space, for example at $\rho = 24.5$. Whereas the efforts for stabilization of this system with the use of a SISO DFC have not been successful

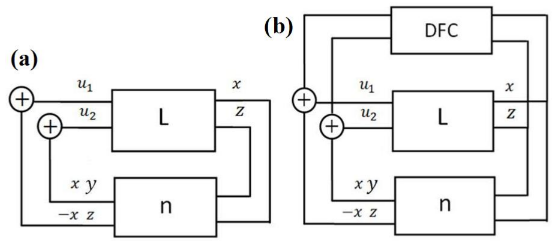

Fig. 2. (a) MIMO Lure form of the Lorenz system. (b) The closed loop system.

up to now, in this paper we consider the system as a MIMO system with the Lure form shown in Fig.2a; which L and n denote respectively the dual-input dual-output linear and nonlinear parts of the system.

$$n = \begin{bmatrix} -zx \\ xy \end{bmatrix} \tag{11}$$

$$L : \begin{cases} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -\mu \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \end{cases}$$

$$\Longrightarrow L(s) = \begin{bmatrix} \frac{\sigma}{s^2+(1+\sigma)s+\sigma(1-\rho)} & 0 \\ 0 & \frac{1}{s+\mu} \end{bmatrix} \tag{12}$$

The goal is to design a MIMO DFC to stabilize an unstable periodic solution of the system. The closed loop system is shown in Fig.2b. So the MIMO DFC will be in the following form

$$\underline{U} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} x(t-\tau) - x(t) \\ z(t-\tau) - z(t) \end{bmatrix} \tag{13}$$

The aim is to determine the gain matrix and delay (τ) of the controller, so that the closed loop system has a periodic response in the form of Eq.3 for ($\rho = 24.5$). For simplicity we consider a simple case that is in the following form

$$\underline{U} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & k \\ 0 & k \end{bmatrix} \begin{bmatrix} x(t-\tau) - x(t) \\ z(t-\tau) - z(t) \end{bmatrix} \tag{14}$$

So our suggested closed loop system is as follows

$$\begin{cases} \dot{x} = -\sigma x + \sigma y \\ \dot{y} = \rho x - y - zx + k(z(t-\tau) - z(t)) \\ \dot{z} = -\mu z + xy + k(z(t-\tau) - z(t)) \end{cases} \tag{15}$$

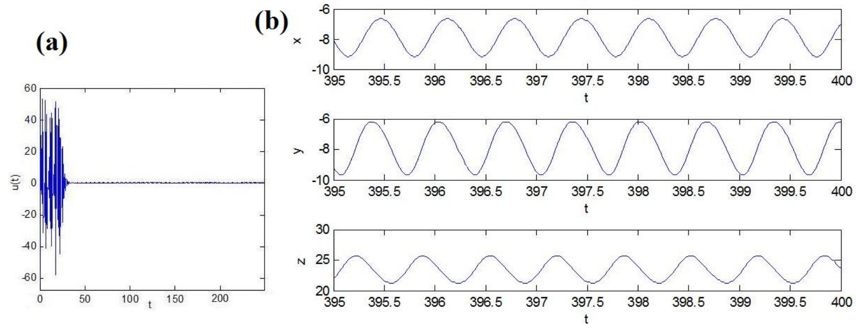

Fig. 3. (a) Control signal tends to zero. (b) The steady state periodic responses of the closed loop system.

τ coincides with the period of the desired periodic solution of the closed loop system. In case stabilization be successful, the control signal will vanish and there will not be any power dissipation in the feedback loop. So by setting $\tau = 2\pi/\omega$ and using the approximation $e^{-\tau s} \approx 1 - \tau s$, we try to determine k. Once more looking back to Eq.3, after doing some calculations similar to those of section (2.1) and substituting A, B and ω with their values at ($\rho = 24.5$) from Eqs. 6-8 ($A = 7.8685, B = 1.0405, \omega = 9.5104$ rad/s $\Rightarrow T = 0.6607$ s), the controller's gain $k = 2.5227$ is obtained. The value obtained for delay here ($T = \tau$) is nearly equal to the value obtained for it in [7] that was 0.67 s.

Fig.3a and Fig.3b show the control signal and the steady state stable periodic responses of the closed loop system. The control signal tends to zero which means that the stabilization strategy has been successful.

Fig.4 shows a zoom view of the steady response of the first state variable of the system ($x(t)$). It illustrates that the bias A, amplitude B and period T of x are equal to those obtained from Eqs. 6-8 at $\rho = 24.5$ and confirms the accuracy of the implemented analytical approach. A noticeable point about

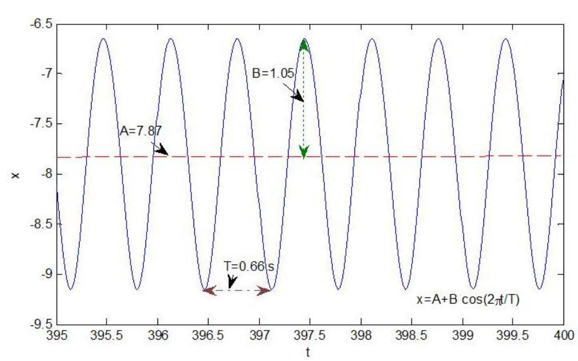

Fig. 4. The bias, amplitude and period of the state variable x.

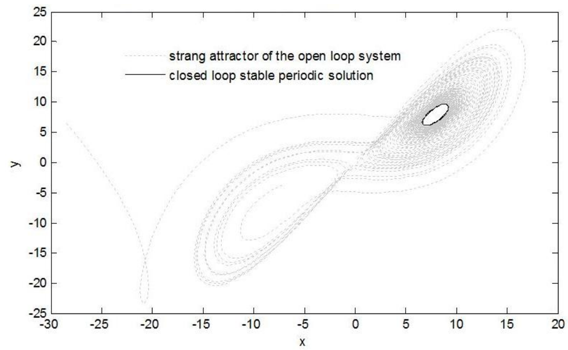

Fig. 5. Stabilizing an unstable periodic orbit embedded in the strange attractor of the open loop system.

the closed loop system is that we have stabilized one of the unstable periodic solutions of the open loop system. In other words we have selected one of the infinite number unstable periodic orbits embedded in the strange attractor and stabilized the system towards it. This fact is shown in Fig.5.

The implemented control structure can be also used for stabilizing the EPs of the Lorenz system. As shown in Fig.6, using $k = 2.5$ and $\tau = 0.8$, the closed loop system will be stabilized to C^+ with the stable eigen values $\lambda_{1,2} = -3.7987 \pm j3.84365$ and $\lambda_3 = -1.561087$.

5 Conclusion

In our paper, using straightforward calculations, the Lorenz system's periodic solutions have been analytically calculated. The results are approximate, due to the first harmonic analysis carried out on the system. The purpose of the paper is to design a DFC in order to stabilize unstable periodic solutions of

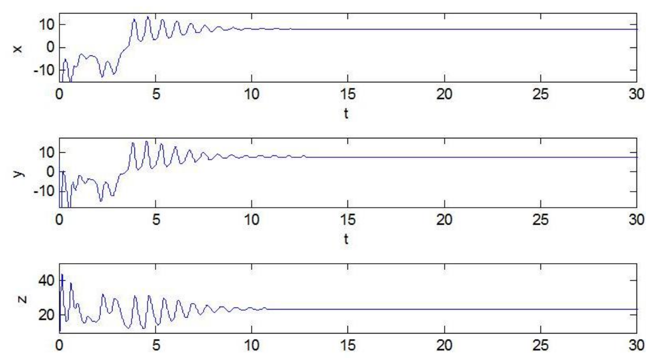

Fig. 6. Lorenz system stabilization to its EPs C^+.

the system. For $\rho = 24.5$ the open loop system shows chaotic behaviors. But the implemented MIMO DFC stabilizes an unstable periodic solution of the system at this value. The key point of our controller, which makes it able to stabilize Lorenz system, is using a different signal in the feedback loop and applying the control signal to the appropriate points. It is the first time, that a simple DFC is implemented successfully for Lorenz periodic solutions stabilization. The simulation results confirm the accuracy of the implemented analytical approach.

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