have been proposed for uncertainty quanti cation. All of these techniques have positive and negative features, and no single technique is optimum for all situations. Following our previous work on UQ (Dinescu *et al.*[1] and Wang *et al.* [7]), here we employed Polynomial Chaos (PC) approach to model uncertainty propagation. Polynomial chaos methods have been successfully applied to solid mechanics problems by several researches (See for example Ghanem and Spanos [3] and Doostan *et al.* [2]). PC schemes have also been employed for a number of uid mechanics problems by a number of researchers such as: Walters and Huyse [6], Mathelin *et al.* [4] and Dinescu *et al.* [1]. The polynomial chaos representation can be used for di erent Probability Density Functions (PDFs) and can be implemented through either intrusive or non-intrusive methods. The intrusive approach requires the modi cation of the CFD codes and this may be di cult, expensive, and time consuming for many CFD problems. Moreover, the source codes of most commercial CFD softwares are not accessible and thus it is impossible to implement the intrusive PC approach to such softwares. For these reasons, here we focused on non-intrusive PC methodology with uniform PDF for uncertainty quanti cation. The main shortcoming of all PC methods is the *curse of dimensionality*. Developing e cient reduced-order models for shortening the computational cost associated with the stochastic analysis is of great interest for prediction of complex industrial ows with large number of uncertain parameters. In recent years, several model reduction techniques have been proposed for uncertainty quanti cation. Two informative examples of such works are: Nouy [5] and Doostan *et al.* [2]. In Nouy [5] a Generalize Spectral Decomposition (GSD) was proposed that gives the reduced basis independent of the stochastic discretization scheme. The GSD implementation to a class of Stochastic Partial Di erential Equations (SPDE) leads to drastic computational saving though does not circumvent the curse of dimensionality. Doostan *et al.* [2] proposed an intrusive model reduction technique for chaos representation of a SPDE to tackle the curse of dimensionality. A 2D test case from solid mechanics is chosen to illustrate the accuracy and convergence of the model.

In this work, a non-intrusive reduced-order technique is developed and applied to the 2D steady-state stochastic heat di usion equation. This paper is organized as follows. In Section 2 we present the details of mathematical formulation and problem under investigation. In Section 3, the model reduction methodology is described. Finally, in Section 4 the numerical results are presented and discussed.

2 Mathematical Formulation

To demonstrate the non-intrusive stochastic model reduction algorithm, 2D steady-state stochastic heat conduction in a square plate of side 2 is considered (see Figure 1). The 2D heat di usion with random thermal conductivity is described by the following SPDE:

$$\frac{\partial}{\partial x}(k(x,y;\zeta)\frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(k(x,y;\zeta)\frac{\partial T}{\partial y}) = 0 \qquad (1)$$

As shown in Figure (1), the top boundary of the plate is at hot temperature

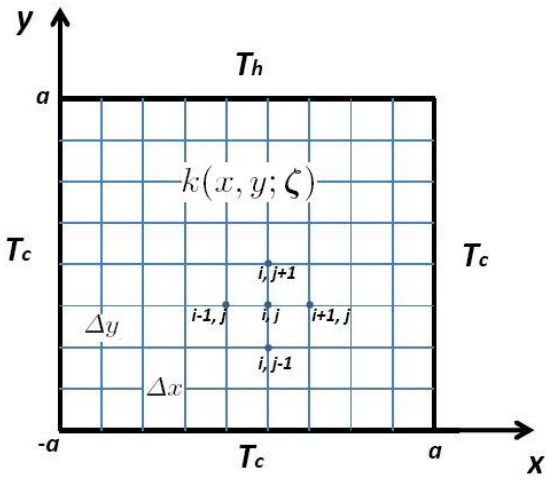

Fig. 1. Schematic of computational domain.

T_h whilst the side and bottom boundaries of the plate are at cold temperature T_c. The thermal conductivity of the plate, $k(x, y; \zeta)$, is assumed to be a two-dimensional homogeneous random process with known mean $k(x, y)$ and known covariance function:

$$R(x_1, y_1; x_2, y_2) = \sigma_k^2 e^{-|x_1 - x_2|/b_x - |y_1 - y_2|/b_y} \quad (2)$$

where b_x and b_y are the correlation lengths in x and y directions, respectively, and σ_k is the standard deviation on the thermal conductivity.

A key ingredient here is the representation of stochastic thermal conductivity field as a Karhunen-Loeve (KL) expansion, a type of Fourier expansion for random functions, which amounts to a discretization in the space of random events. According to the KL expansion, the eigenvalues and eigenfunctions are obtained by solving the following 2D integral equation:

$$\int_D R(x_1, y_1; x_2, y_2)\phi_n(x_2, y_2)dx_2 dy_2 = \lambda_n \phi_n(x_1, y_1) \quad (3)$$

Separation of kernel (2) as $R(x_1, y_1; x_2, y_2) = \sigma_k^2 e^{-|x_1 - x_2|/b_x} . e^{-|y_1 - y_2|/b_y}$ and substitution in (3) leads to two identical 1D integral eigenvalue equations in x and y directions. Solution of the integral equations give eigenvalues (i.e. $\lambda_i^{(x)}$ and $\lambda_j^{(y)}$) and their corresponding eigenfunctions (i.e. $\phi_i^{(x)}$ and $\phi_j^{(y)}$). As described in Ghanem and Spanos [3], the complete form of KL expansion for random process $k(x, y; \zeta)$ is:

$$k(x, y; \zeta) = k(x, y) + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left\{\sqrt{\lambda_i^{(x)}\lambda_j^{(y)}}\zeta_{i,j}\ \frac{1}{\sqrt{2}}[\phi_i^{(x)}(x)\phi_j^{(y)}(y) + \phi_j^{(x)}(x)\phi_i^{(y)}(y)]\right\} \quad (4)$$

Having obtained an analytical expression for the thermal conductivity, the SPDE (Equation (1)) is discretized using an explicit central di erencing scheme in a uniform grid ($\Delta x = \Delta y$), see Figure 1. Thus, for any set of $\boldsymbol{\zeta} \equiv \{\zeta_i\}_{i=1}^{n}$, rst thermal conductivity is calculated in the computational domain using KL expression (Equation (4)). Then, the new temperature T^{n+1} at grid node (i, j) is obtained from old nodal temperature T^n of neighbouring nodes. The solution is converged when the maximum error between the old and new temperature values is su ciently small($\epsilon \simeq 10^{-9}$).

3 Model Reduction Methodology

In the classical polynomial chaos expansion, the random temperature eld $T(x, y; \boldsymbol{\zeta})$ can be decomposed into deterministic and stochastic components. The PC representation of temperature eld of order p for n random variable $\boldsymbol{\zeta} \equiv \{\zeta_i\}_{i=1}^{n}$ can be written as:

$$T(x, y; \boldsymbol{\zeta}) - < T(x, y) > = \sum_{i=1}^{P} T^i(x, y)\psi_i(\boldsymbol{\zeta}) \tag{5}$$

where the total number of terms are $P + 1 = (p + n)!/p!n!$ and the mean value of $T(x, y; \boldsymbol{\zeta})$ is expressed as:

$$< T(x, y) > = \int_{!} T(x, y; \boldsymbol{\zeta}) f(\boldsymbol{\zeta}) d\boldsymbol{\zeta} \tag{6}$$

In the above equation, f is Probability Density Function (PDF). Here we assumed random variables are uniformly distributed over interval [-1,1] and thus the PDF is $f = 1/2^n$ for n random variables $\{\zeta_i\}_{i=1}^{n}$. The non-intrusive method uses spectral projection to nd the PC expansion coe cients $T^i(x, y)$ in Equation (5). Projecting Equation (5) onto the k^{th} basis and use of orthogonality gives:

$$T^i(x, y) = \frac{1}{< \psi_i^2(\boldsymbol{\zeta}) >} \int_{!} T(x, y; \boldsymbol{\zeta}) \psi_i(\boldsymbol{\zeta}) f(\boldsymbol{\zeta}) d\boldsymbol{\zeta} \tag{7}$$

The objective of the spectral projection method is to compute the polynomial coe cients by evaluating numerator in Equation (7) numerically, while the dominator can be computed analytically for multi-variant orthogonal polynomials. Here we used the n-dimensional $Gauss\text{-}Legendre\ quadrature$ to compute the projection integrals in Equation (7) as:

$$T^i(x, y) = \frac{1}{< \psi_i^2(\boldsymbol{\zeta}) >} \sum_{i_1=1}^{q} .. \sum_{i_n=1}^{q} (w_1^{i_1} \otimes .. \otimes w_n^{i_n}) T(x, y; \zeta_1^{i_1}, .., \zeta_n^{i_n}) f(\zeta_1^{i_1}, .., \zeta_n^{i_n}) \tag{8}$$

where $(\boldsymbol{\zeta}^k, \boldsymbol{w}^k)$, $k = 1, 2, .., q$ are the one-dimensional (1D) Gauss-Legendre integration points and weights.

The above classical expansion dose not represent an optimal PC representation of $T(x, y, \zeta)$. To nd the optimal PC expansion one can consider the fact that spatial discretization errors and random discretization errors may be decoupled. Therefore, one can minimize the random discretization errors on the coarse grid and then solve the real physical problem on a ne mesh by using limited number of optimal random basis $\{z_i\}_{i=1}^{m}$ (obtained in the coarse grid analysis) where m is the number of dominated eigenvalues. The rst step in the model reduction scheme is to nd optimal PC basis using POD; a well-known procedure for extracting a basis for a model decomposition from an ensemble of realizations. To this end, suppose in a coarse grid, expression (9) represents an optimal PC expansion of the stochastic temperature eld $T(x, y, \zeta)$;

$$T(x, y; \zeta) - < T(x, y) > = \sum_{i=1}^{m} T^i(x, y) z_i(\zeta) \tag{9}$$

Now in the coarse grid, the covariance function $C(x_1, y_1; x_2, y_2)$ of temperature eld can be obtained from:

$$C(x_1, y_1; x_2, y_2) = \sum_{i=1}^{P} T^i(x_1, y_1) T^i(x_2, y_2) < \psi_i^2 > \tag{10}$$

The corresponding eigenvalues ν_i and eigenfunctions $\phi_i(x, y)$ are the solution of the following eigenvalue problem:

$$\int_D C(x_1, y_1; x_2, y_2) \phi_i(x_2, y_2) dx_2 dy_2 = \nu_i \phi_i(x_1, y_1) \tag{11}$$

The upper limit m in the Equation (9) can be found by the size of dominant eigenspace (10) such that $\sum_{i=1}^{m} \nu_i / \sum_i \nu_i \geq 0.99$.
Having obtained $T^i(x, y)$ from classical PC on the coarse grid and eigenfunctions $\phi_i(x, y)$ from the solution of eigenvalue problem (11), the set of optimal basis $\{z_i\}_{i=1}^{m}$ can be now represented as a linear combination of the set of classical polynomial chaos, $\{\psi_i\}_{i=1}^{P}$ using the following scalar product[1]:

$$z_i(\zeta) = [\, T(x, y; \zeta) - < T(x, y) >, \phi_i(x, y)] = \sum_{j=1}^{P} \alpha_{ij} \, \psi_j(\zeta) \tag{12}$$

where coe cient α_{ij} are obtained via the scalar product:

$$\alpha_{ij} = \int_R T^j(x, y) \phi_i(x, y) d\vec{x} d\vec{y} \tag{13}$$

One now dose the classical polynomial chaos on a ne mesh, where z_i are used instead of ψ_i. For $1 \leq i \leq m$, the coe cients in expansion (9) are obtained from:

$$T^i(x, y) = \frac{< T(x, y; \zeta), z_i >}{< z_i, z_i >} = \frac{1}{\nu_i} \sum_{j=1}^{P} \alpha_{ij} \; < T(x, y; \zeta), \psi_j > \tag{14}$$

[1] The scalar product of functions v and w is de ned as: $[v, w] = \int_x v.w dx$.

4 Results and Discussion

We now examine the performance of the reduced-order model by analyzing the 2D steady-state heat conduction equation. It is assumed that the top wall is at T_h = 300°C and side and bottom walls at T_c = 100°C. First, a 2D KL expansion is performed using the exponential kernel with a standard deviation of σ_k = 1.0 $W/m.K$ and correlation lengths of b_x = b_y = 10.0 m. The mean thermal conductivity is assumed to be k = 5.0 $W/m.K$. The first six largest terms in the KL expansions are chosen for further analysis.

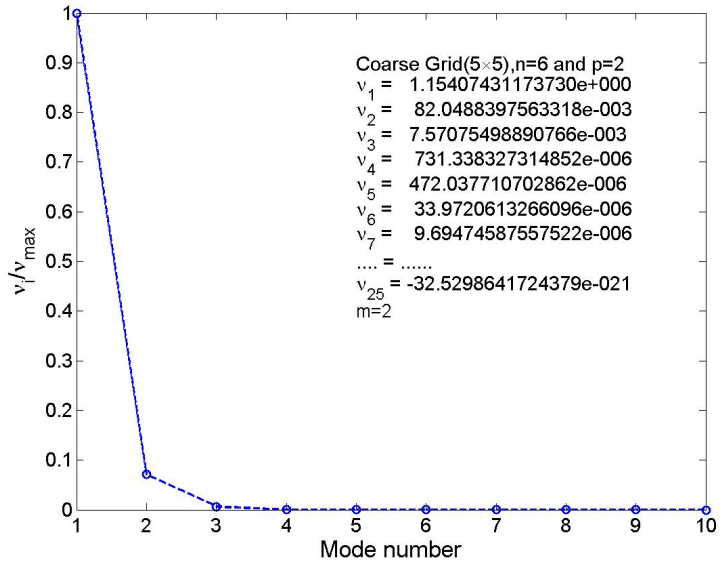

Fig. 2. Computed eigenvalues using coarse discretization analysis.

Figures 2 and 3 respectively show the distributions of eigenvalues and $|\alpha_{ij}|$ coefficients obtained from the coarse discretization analysis on a 5 × 5 mesh when a second-order Legendre polynomial (p = 2) is employed. From these figures it can be concluded that only two (m = 2) basis functions (i.e. z_1 and z_2) are adequate for the fine discretization analysis. Thus, fine discretization analysis is performed using the new z_1 and z_2 basis functions on a 40 × 40 mesh. The computed mean and variance fields using full- and reduced-order models are compared in Figure 4. It is visible the fine grid computations via reduced- and full-order models resulted in identical results for the mean temperature. Moreover, full- and reduced-order analysis on the fine mesh produced very similar variance fields. The order of maximum difference in variance fields is 10^{-3}. The ratio of computation-time for the reduced-order analysis to the time needed for the full-order calculation using three fine meshes of 30 × 30, 40 × 40 and 50 × 50 is shown in Figure 5. While reduced-order computations on the 30 × 30 results to about 10% saving in the computation-time, more that 50% saving in computation-time is obtained when a finer 50 × 50 mesh is employed.

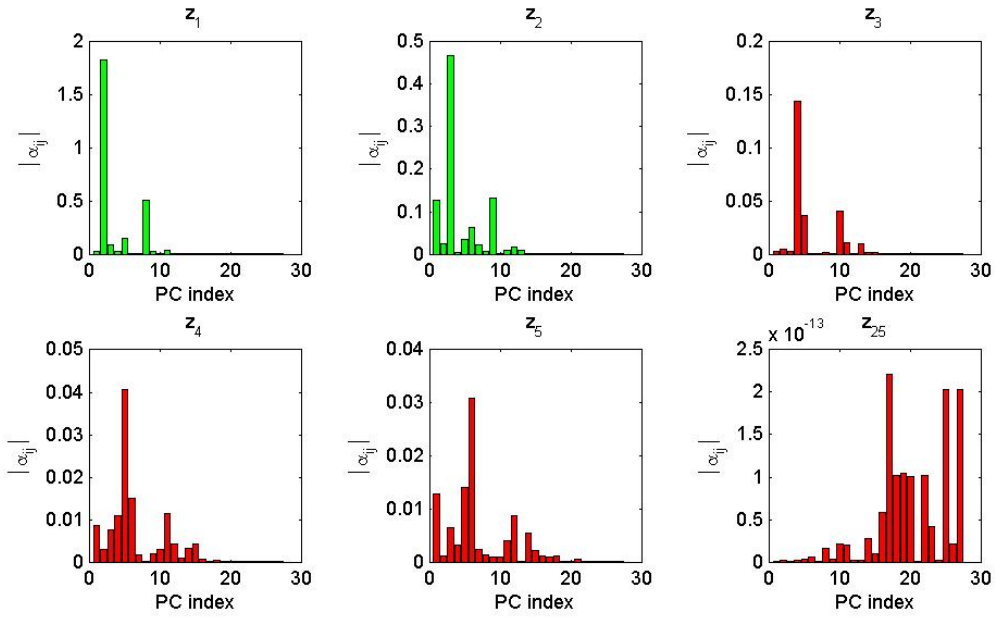

Fig. 3. Computed $|\alpha_{ij}|$ using coarse discretization analysis.

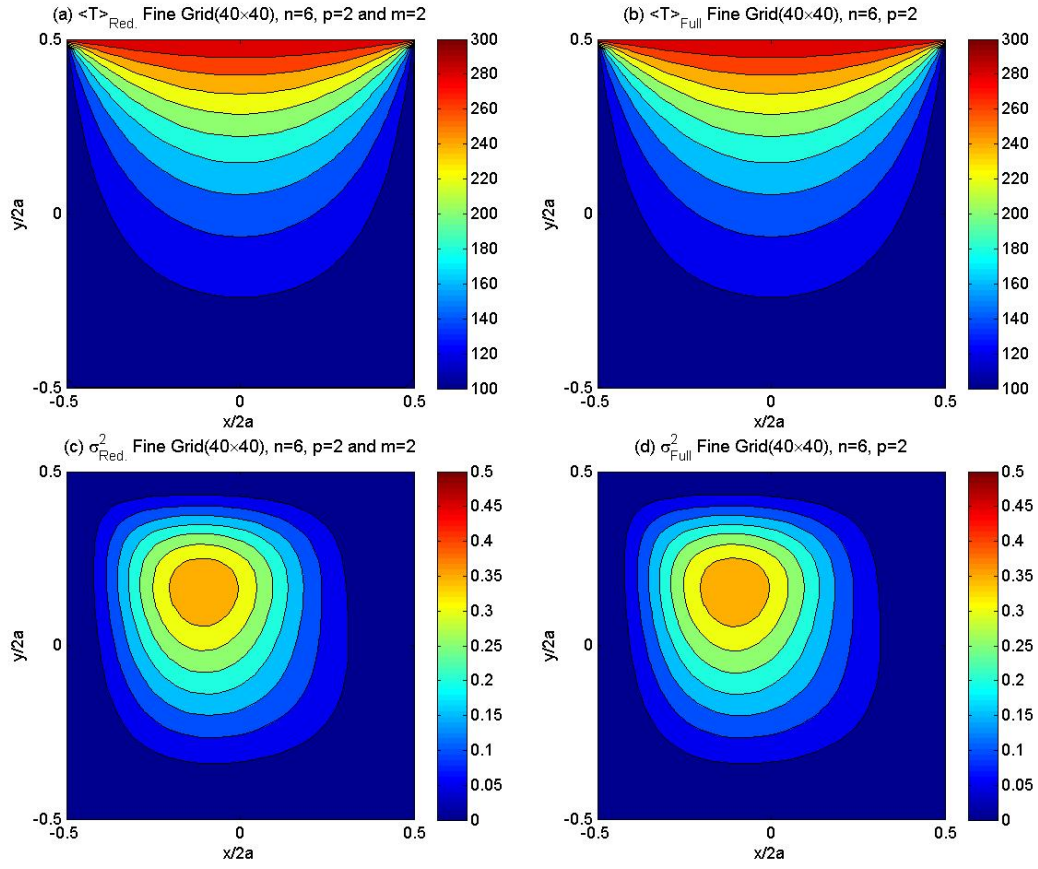

Fig. 4. Comparison of mean and variance elds.

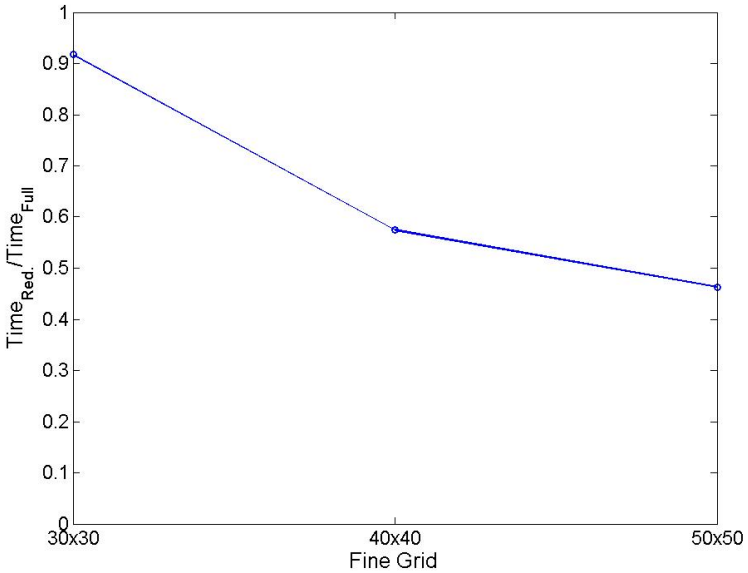

Fig. 5. Ratio of reduced-order computation-time to the full-order computation-time.

Conclusion

In this paper, a non-intrusive model reduction technique for PC expansion is presented and discussed. The reduced-order model is applied to the 2D steady-state heat di usion equation. Distributions of mean and variance obtained from the reduced-order model are compared with those of full-order model. The numerical results show that the developed reduced-order model is able to produce acceptable results for such statistical quantities. Computation-time of the reduced-order model is found to be lower than that of the full-order model.

References

1.C. Dinescu, S. Smirnov, Ch. Hirsch, and C. Lacor. Assessment of intrusive and non-intrusive non-deterministic CFD methodologies based on polynomial chaos expansions. *Int. J. of Eng. Systems Modeling and Simulation*, 2:87-98, 2010.

2.A. Doostan, R. Ghanem, and J. Red-Horse. Stochastic model reduction for chaos representations. *Comp. Meth. in Appl. Mech. and Eng.*, 196:3951-3966, 2007.

3.R. Ghanem, and O. Spanos. Stochastic Finite Elements: A Spectral Approach, 1991, Springer Verlag.

4.L. Mathelin, M. Y. Hussaini, and T. A. Zang. Stochastic approaches to uncertainty quanti cation in CFD simulations. *Numer. Algorithms*, 38:209-236, 2007.

5.A. Nouy. A generalized spectral decomposition technique to solve a class of linear stochastic partial di erential equations. *Comp. Meth. in Appl. Mech. and Eng.*, 196:4521-4537, 2007.

6.R. W. Walters, and L. Huyse. Uncertainty analysis for uid mechanics with applications. NASA/CR 2002-211449.

7.X. Wang, Ch. Hirsch, Z. Liu, S. Kang, and C. Lacor. Uncertainty-based robust aerodynamic optimization of rotor blades. *Int. J. Numer. Meth. Engng*, DOI: 10.1002/nme.4438, 2012.